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MODELING THE TERM STRUCTURE OF INTEREST RATES: WHERE DO WE STAND?

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The views expressed in this paper are those of the author and do not necessarily reflect the views of the National Bank of Belgium.

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Abstract

No-arbitrage term structure models are becoming increasingly important to policy makers and practitioners alike. Several factors justify this trend. First, modeling progress has been tremendous over the last years, allowing a much better fit of actual yield curve dynamics and increased model realism (see Dai and Singleton (2002a,b)). Second, increases in computing power allow the efficient panel estimation of term structure models. Given that term structure models have implications for both the cross-section and time series dimension of yields, panel estimation techniques are to be preferred over either cross-section or time series techniques. Third, term structure models have recently been extended in ways that are of direct interest to policy makers. Example given, Dewachter and Maes (2001) model the international term structure of interest rates, taking into account the role of the exchange rate in a no-arbitrage economy, while amongst others Hördahl et al. (2002) and Dewachter et al. (2002) *jointly* model the term structure of interest rates with the dynamics of macroeconomic variables. The latter approach allows to study (*i*) the driving factors behind the term structure and the risk premia in terms of clearly interpretable macroeconomic variables and their determinants, and (*ii*) the effects of monetary policy on the term structure of interest rates and macroeconomic variables within a consistent no-arbitrage framework.

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1 Introduction

"The quest for understanding what moves bond yields has produced an enormous literature with its own journals and graduate courses. Those who want to join the quest are faced with considerable obstacles. The literature has evolved mostly in continuous time, where stochastic calculus reigns and partial differential equations spit fire. The knights in this literature are fighting for different goals, which makes it often difficult to comprehend why the quest is moving in certain directions. But the quest is moving fast, and dragons are being defeated."

- Monika Piazzesi (2002) -

The term structure of interest rates has intrigued and fascinated generations of academic researchers and practitioners. This should not come as a surprise. An understanding of the stochastic behavior of yields is important for the conduct of monetary policy, the financing of public debt, the formation of expectations about real economic activity and inflation, the risk management of a portfolio of securities, and the valuation of interest rate derivatives. The term structure of interest rates is forward looking by construction and encapsulates both market expectations and the expected excess returns (or *risk premia*) that are required by investors as compensation for being exposed to different sources of macroeconomic risk[1].

Interest is payment (received) for borrowing (lending) funds over a period of time. An interest rate is the amount of interest per unit of time as a fraction of the total amount of funds. The *yield* of a particular bond is the fictional, constant, known, annual interest rate that equates the price of the bond to the sum of the present values of its generated cash flows, assuming that the bond issuer does not default. Put differently, the yield is a precisely defined measure of the average return offered to the bondholder over the remaining time to maturity of the bond. By construction, there is a one-to-one relationship between yields and bond prices, so studying yields is equivalent to studying bond valuation. We define a *zero coupon bond* (ZCB) as a claim that does not promise any payments (coupons) during its time to maturity, but promises to pay 1 unit of account (or equivalently 100% of the face value) at its time of maturity. This paper focuses attention to both the time series and cross-section of yields that are implied by default-risk-free ZCBs. The time t ZCB *term structure of interest rates*, or *yield curve*, is the curve that arises when we plot the yields at time t of ZCBs that mature at increasing

[1] In general, the price of *any* financial asset is determined by the expected value of its future (un)certain cash flows. Hence, the price reflects (rational or irrational) market expectations as well as investors' attitudes towards the risks they are exposed to. The term structure will be particularly well-suited to teach us something about market expectations. By filtering out the risk premia, market expectations can be isolated and analyzed.

times to maturity.[2] In practice, the majority of existing bonds pay an annual or semi-annual coupon, are issued by governments and corporations, and often come with options attached to them. So, studying yields (prices) of default-risk-free ZCBs might seem uninterestingly restrictive at first sight. However, coupon bonds can be considered to be portfolios of ZCBs with payoffs and maturities that match the coupon payments. In the same way, callable or putable coupon bonds can be regarded to be portfolios of discount bonds and plain options. Finally, the modern modeling approach to value bonds with credit risk is similar to the one presented here (see e.g. Duffie and Singleton (1999)).

A *model* of the *term structure of interest rates* makes explicit how the yields of ZCBs, differing only in their time to maturity, relate to each other at each given point in time. The easiest approach to model the term structure is by making use of a purely statistical model. Example given, one might perform a factor analysis or principal component analysis of yield changes and express the covariance matrix of yields in terms of a few factors that describe their common movement (see for example Bliss (1997)). Alternatively, regulators often use techniques that fit the empirical term structure with a specific functional form (e.g. Nelson and Siegel (1987) and Svensson (1994)). The problem with all these approaches is that one can quite easily reach a statistical representation of yields that implies an arbitrage opportunity[3]. Hence, these models are ill-suited for the economic understanding of yields and for the applications mentioned in the first paragraph above.

This paper introduces the reader to the modeling of the term structure of interest rates. In essence, all the so-called term structure models are driven by the assumption that arbitrage opportunities are absent. The intuitive concept of *absence of arbitrage* can be linked directly to the existence of a pricing kernel and a risk neutral probability measure. The latter concepts are at the heart of the modern finance literature and play a unifying role in it (see Cochrane (2001) and Duffie (2001)). Researchers have developed a multitude of term structure models (TSMs) that are consistent with the absence of arbitrage opportunities during the $1980s$ and $1990s$[4]. It turns out that imposing absence of arbitrage opportunities

[2] Of course, zero coupon bonds with the same term to maturity might still have different yields due to differences in default or credit risk, liquidity risk, and income tax rules. The resulting relationship among these yields for a given time to maturity is refererred to as the *risk* structure of interest rates.

Here, we will be concerned solely with the *term* structure of yields on *default-risk-free* zero coupon bonds. The qualifier "default-risk-free" will be left away in the following since default risk is considered nowhere in the paper.

[3] Example given, statistical analyses such as in Litterman and Scheinkman (1988) usually suggest that a "level" factor, shifting all yields up or down, accounts for a significant part of the yield variability. However, Dybvig and Ingersoll (1996) derive that, in order to preclude arbitrage opportunities, the long-maturity yield must converge to a constant.

For a discussion of the inconsistencies in the Nelson-Siegel techniques, see Filipovic (2000) and Brousseau (2002).

[4] For treatments with differing degrees of formalism, consult the review papers of Back (1996), Bolder (2001), Chapman and Pearson (2001), Dai and Singleton ($2002b, 2002c$), Fisher (2001),

in financial markets implies strong restrictions on the time series behavior of each yield and on the cross-section of yields at each point in time. By all means, these models have made a substantial impact in the financial services industry, proving that sophisticated finance theory can be of practical use.

The paper is organized in the following way. Section 2 briefly sketches how probability spaces and probability models are used in the finance literature to describe real world uncertainty. Section 3 studies absence of arbitrage and how this central concept translates into the main pricing equations. The main theoretical building blocks that play a unifying role in modern finance are intuitively discussed here.[5] In section 4, we focus attention to the popular affine class of factor TSMs and show explicitly how we can quasi-analytically solve for the ZCB prices when making specific assumptions about the underlying stochastic factors and the so-called prices of risk (or instantaneous expected reward to risk ratios). Moreover, the strengths and weaknesses of a tractable class of mathematical term structure models are discussed and the alternative models that have been put forward recently are reviewed. Section 5 concludes.

2 Modeling Uncertainty

Financial economics is about how people allocate scarce resources over time and under uncertainty. This section discusses in an informal way how the finance literature approaches (models) uncertainty. The cash flows generated by financial assets may be known with certainty, but at least the discount rates in the future, *i.e.* future interest rates, are contingent on the uncertain future state of the economy. The economy could be booming in the future, which typically gives rise to relatively high interest rates, or a future recession could result in relatively low interest rates. Hence, from the perspective of what is known with certainty today, attaching a value to even default-risk-free ZCBs will involve dealing with uncertainty in some way. During the last two decades, the modeling approach to asset pricing under uncertainty has been streamlined to a more or less unified approach around the notion of *absence of arbitrage opportunities*, or equivalently, around the existence of a *pricing kernel* or a *risk neutral probability measure.*

2.1 Probability Spaces

To represent uncertainty, we choose a *probability space* $(\Omega, \mathcal{F}, \mathcal{P})$, on which all stochastic processes in this paper will be defined. Informally, the *sample space* Ω

Gibson, et al. (2001), Lund (1998), Piazzesi (2002), Rogers (1995), Subrahmanyam (1996), Sundaresan (2000), and Yan (2001), or books such as Cochrane (2001), Dothan (1990), Duffie (2001), Karatzas and Shreve (1988, 1991), Mikosch (1999), Neftci (1996), and Shimko (1992).

[5] To maximize intuition and understanding and minimize on notational burden, we set out the theory in sections 2 and 3 in univariate calculus. Multivariate calculus does not pose any conceptual problems, however, and we will make use of it in section 4.

unites all possible outcomes of the state of the world. Subsets of Ω are referred to as events. A *tribe* (or *σ-algebra*) $\mathcal{F}$ is a collection of "interesting" events. Two extreme cases of a tribe include the minimal one, $\mathcal{F}_0 = \{\phi, \Omega\}$, where ϕ represents the empty set, and the maximal one, $\mathcal{F}_\infty$, which has as elements all possible subsets of Ω (sometimes referred to as the power set). For our purposes, we are interested in sequences of events across time, and we will make Ω a set with increasing dimensions over time. We create a *filtration* $\mathbb{F}$, which is a sequence of tribes, $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t \subset \mathcal{F}_\infty$, representing the information of agents through time. A filtration can be intuitively understood as an ever increasing stream of information, fixing a history of events. By assumption, agents do not forget, implying that $\mathcal{F}_s \subset \mathcal{F}_t$, whenever $s \leq t$. In the limit, complete information is obtained, $\mathcal{F}_\infty$. A *probability measure* $\mathcal{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathcal{P} : \mathcal{F}_t \rightarrow [0, 1]$, satisfying $\mathcal{P}(\phi) = 0$ and $\mathcal{P}(\Omega) = 1$. The function $\mathcal{P}$ assigns a number (probability) to every set $F \in \mathcal{F}_t$.

Armed with a probability space triple $(\Omega, \mathcal{F}, \mathcal{P})$, we can define a *random variable* X as a mapping from (subsets of) Ω to $\mathbb{R}$. For example, the monthly return of a ZCB is a random variable. It can be understood as a black box that takes qualitative events as inputs and gives a real number, the ZCB return, as an output. Random events most often have qualitative aspects. A random variable gives a quantitative flavor to those qualitative events, allowing them to be handled effectively. A continuous *stochastic process* $\{X(t)\}_{t\geq 0}$ (or simply $X(t)$) is a collection of random variables. Both a random variable and a stochastic process have random realizations, but the realization of a random variable is a number in $\mathbb{R}$, whereas the realization of a stochastic process is a function on the time domain. Put differently, a stochastic process is random in terms of the trajectory as a whole, rather than in terms of a particular value at a specific point in time. Intuitively, it represents a random drawing from a collection of possible trajectories. Choosing a certain state of the world determines the complete trajectory over time.

A *conditional expectation* operator, $E^{\mathcal{P}}[\cdot| \mathcal{F}_t] \equiv E_t^{\mathcal{P}}[\cdot]$, is *always* defined with respect to a certain information set, say $\mathcal{F}_t$, and with respect to a certain probability measure, say $\mathcal{P}$. Finally, a *$\mathcal{P}$-martingale* is defined as a stochastic process $X(t)$ for which it holds that $E_t^{\mathcal{P}}[X(T)] = X(t)$, for all $T \geq t$. There are two additional technical conditions: $X(t)$ should be adapted to $\mathcal{F}_t$ (meaning that $X(t)$ should be known at time t), and $E[|X(t)|] < \infty$ for all t.

2.2 Standard Brownian Motion

Standard Brownian motion is a stochastic process named after the biologist Robert Brown, who claimed to have observed it in pollen suspended in water around 1827. Much of the scientific work for this model was initially done by Louis Bachelier in 1900 (see Courtault et al. (2000) for a historical account), and later more rigorously elaborated by Albert Einstein (1905), Norbert Wiener (1923), and others. Proving that there exists a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is not obvious, but always possible. When we say that a stochastic process, such as $W(t)$ is a standard Wiener process under a given probability measure $\mathcal{P}$, we are saying that the probabilities of its possible paths are assigned such that its central properties (such as zero expected value) are preserved. For instance, when the probability measure is changed to, say, $\mathcal{Q}$, such that the process no longer has a zero expected value, it is no longer the same thing. But it can be proven that another process, say $W^{\mathcal{Q}}(t)$, does have the property of a standard Wiener process under the new probability measure $\mathcal{Q}$ (the Girsanov-Cameron-Martin theorem).

Consider a discrete-time random walk description:

$$\begin{aligned} &W_{t+\Delta t} = W_t + \varepsilon_{t+\Delta t}, \\ &W_0 = 0,\ \varepsilon_{t+\Delta t} \sim \mathcal{N}(0, \Delta t). \end{aligned} \tag{1}$$

Now consider the continuous-time behavior of this process as Δt tends towards the infinitesimal time increment dt, where dt is heuristically defined as the smallest positive real number such that (Shimko (1992)):

$$dt^{\alpha} \equiv 0 \text{ whenever } \alpha > 1. \tag{2}$$

A standard *Wiener process* or standard *Brownian motion* $W(t)$ can be constructed by taking the limit of the discrete-time random walk (1) with jumps generated by a standard normal variable with mean zero and variance equal to the time between jumps, as the time between jumps converges to dt:

$$\begin{aligned} &W(t+dt) = W(t) + \varepsilon(t+dt), \\ &W(0) = 0,\ \varepsilon(t+dt) \sim \mathcal{N}(0, dt), \end{aligned} \tag{3}$$

and define $dW(t) \equiv W(t+dt) - W(t)$. Both $dW(t)$ and $\varepsilon(t+dt)$ are referred to as white noise. Intuitively, $W(t)$ corresponds to the concept of a continuous-time random walk. Six properties follow by construction[6]:

$$\begin{array}{llll} (i) & E[dW(t)] & = 0, \\ (ii) & E[dW(t)dt] & = dtE[dW(t)] = 0, \\ (iii) & E\left[(dW(t))^2\right] & = dt, \\ (iv) & E\left[(dW(t)dt)^2\right] & = dt^2 E\left[(dW(t))^2\right] = 0, \\ (v) & Var\left[(dW(t))^2\right] & = E\left[(dW(t))^4\right] - \left(E\left[(dW(t))^2\right]\right)^2 \\ & & = 3dt^2 - dt^2 = 0, \\ (vi) & Var[dW(t)dt] & = E\left[(dW(t)dt)^2\right] - E[dW(t)dt]^2 = 0. \end{array} \tag{4}$$

These properties are important because they demonstrate that the variances of $(dW(t))^2$ and $dW(t)dt$ vanish. The expectation operator in (ii) and (iii) becomes

[6] Properties (i), (ii), and (iii) follow from standard statistics and the definition of $dW(t)$ in (3). Property (iv) follows from the heuristic definition of dt in (2). Property (v) follows from the use of the fourth central moment of a standard normally distributed variable and from property (iii). Property (vi) follows from properties (ii) and (iv).

redundant if the corresponding variance is zero (properties (v) and (vi)). Hence, in continuous time, the following heuristic *multiplication rules* are implied by the above properties:

$$\begin{cases} \text{rule 1:} & dW(t)^2 &= dt, \\ \text{rule 2:} & dW(t)dt &= 0, \\ \text{rule 3:} & dt^2 &= 0, \end{cases} \tag{5}$$

where the last rule follows from our initial heuristic definition of dt in (2). The first multiplication rule states that the square of this special stochastic increment is a purely deterministic quantity. It is clear that $dW(t)$ is a special kind of differential, a stochastic differential, not to be confused with the ordinary differentials, say dx, of the Newtonian calculus. Although $dW(t)$ is a random variable, $dW(t)^2$ is not. Hence, in calculating Taylor expansions of functions of $dW(t)$, the second order term is of order dt and must be retained (see below). This unexpected result is the foundation of a new calculus with respect to $W(t)$, *stochastic* or *Itô calculus*.

2.3 Diffusions and Itô's Lemma

A *stochastic differential equation* (SDE) or *diffusion*:

$$dX(t) = \mu_X(X(t), t)\, dt + \sigma_X(X(t), t)\, dW(t), \tag{6}$$

with initial condition $X(0) = X_0$, is a way to model a stochastic process in a continuous-time framework using a standard Wiener process. The solution to the SDE (6) is often referred to as an *Itô process*. Equation (6) always has a strong solution $X(t)$ if some regularity conditions are fulfilled with respect to $\mu_X(X(t), t)$ and $\sigma_X(X(t), t)$ (see Duffie (2001), appendix E). The so-called "growth" conditions make sure that the solution does not explode, while the so-called "Lipschitz" conditions make sure that the solution is unique. A SDE can be understood as an ordinary (or deterministic) differential equation (ODE) that gets perturbed by the arrival of new information, modeled as a standard Wiener process. Intuitively, equation (6) implies that:

$$\begin{aligned} E[\Delta X(t)] &= E[X(t+\Delta t) - X(t)] &&\approx \mu_X(X(t), t)\,\Delta t, \\ Var[\Delta X(t)] &= Var[X(t+\Delta t) - X(t)] &&\approx \sigma_X(X(t), t)^2\,\Delta t, \end{aligned} \tag{7}$$

such that the standard deviation of the change in $X(t)$, $X(t+\Delta t) - X(t)$, is approximately $\sigma_X(X(t), t)\sqrt{\Delta t}$. The drift function $\mu_X(X(t), t)$ accounts for the evolution of the mean in the discrete-time interval $\Delta t > 0$, whereas the volatility function $\sigma_X(X(t), t)$ accounts for the evolution of the standard deviation.

Applying the traditional ODE approach to the solution of a SDE will fail since a Brownian motion can be shown to be nowhere differentiable. The *stochastic integral equation* (SIE) representation of the SDE in (6) is:

$$X(t) = X_0 + \int_0^t \mu_X(X(s), s)\, ds + \int_0^t \sigma_X(X(s), s)\, dW(s), \tag{8}$$

where the first integral over time in equation (8) is a traditional integral in the Stieltjes sense, while the second, integrating over Brownian motion, is a so-called stochastic or Itô integral. The main contribution of Itô (1951) to the theory of stochastic processes lies in the definition of an integral when the integrator is a Brownian motion. The results of this Japanese researcher have proven to be of such importance that the stochastic calculus is often simply referred to as *Itô calculus*.

The most useful result of Itô calculus is *Itô's lemma*, which is a stochastic version of the chain rule in ordinary calculus. In essence, Itô's lemma implies that any twice differentiable function $F(\cdot)$ of an Itô process $X(t)$ is itself an Itô process. The lemma follows from applying a Taylor series expansion to the function $F(\cdot)$:

$$\begin{aligned} F(X(t)+dX(t),t+dt) &= F(X(t),t)+\frac{1}{1!}\frac{\partial F}{\partial t}dt+\frac{1}{2!}\frac{\partial^2 F}{\partial t^2}(dt)^2+... \\ &\quad +\frac{1}{1!}\frac{\partial F}{\partial X}dX(t)+\frac{1}{2!}\frac{\partial^2 F}{\partial X^2}(dX(t))^2+\frac{1}{3!}\frac{\partial^3 F}{\partial X^3}(dX(t))^3+..., \end{aligned} \tag{9}$$

and then invoking the multiplication rules in (5). We finally get the following equation:

$$dF(X(t),t)=\frac{\partial F}{\partial t}dt+\frac{\partial F}{\partial X}dX(t)+\frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX(t))^2. \tag{10}$$

The last term in (10) drops away in Newtonian calculus, yielding the familiar chain rule result, since any deterministic differential in ordinary calculus is considered to have a limit of zero if raised to a power greater than 1. In stochastic calculus, $(dX(t))^2$ does not vanish due to the finiteness of the quadratic variation of standard Brownian motion (multiplication rule 1 in (5)), even though higher order terms still vanish. Substituting the SDE (6) in (10) and again invoking the multiplication rules in (5), we see that $dF(X(t),t)$ has drift $\partial F/\partial t+\mu_X(\cdot)(\partial F/\partial X)+1/2\sigma_X(\cdot)^2(\partial^2 F/\partial X^2)$ and volatility $\sigma_X(\cdot)(\partial F/\partial X)$, or:

$$\begin{aligned} &dX(t)=\mu_X(\cdot)\,dt+\sigma_X(\cdot)\,dW(t), \\ &dF(X(t),t)=\left(\frac{\partial F}{\partial t}+\mu_X(\cdot)\frac{\partial F}{\partial X}+\frac{1}{2}\sigma_X(\cdot)^2\frac{\partial^2 F}{\partial X^2}\right)dt+\left(\sigma_X(\cdot)\frac{\partial F}{\partial X}\right)dW(t). \end{aligned} \tag{11}$$

The above application of Itô's lemma expresses the volatility and the drift of the function of a stochastic process in terms of the volatility and drift of the process itself and the derivatives of the function. This is the simplest version of Itô's lemma. It can be extended by assuming that the function $F(\cdot)$ is a function of a $K\times 1$ vector $X(t)$ and t, $F(X(t),t)$. Correspondingly, in equation (11), $dX(t)$, $\mu_X(\cdot)$, and $dW(t)$ are $K\times 1$ vectors, and $\sigma_X(\cdot)$ is a $K\times K$ matrix. The multivariate version of Itô's lemma can be written as:

$$\begin{aligned} dF(X(t),t)=&\left(\frac{\partial F}{\partial t}+\mu_X(\cdot)'\frac{\partial F}{\partial X}+\frac{1}{2}tr\left[\sigma_X(\cdot)\sigma_X(\cdot)'\frac{\partial^2 F}{\partial X\partial X'}\right]\right)dt \\ &+\left(\frac{\partial F'}{\partial X}\sigma_X(\cdot)\right)dW(t), \end{aligned} \tag{12}$$

where $tr[\cdot]$ stands for the trace of the matrix between squared brackets.

2.4 Modeling Interest Rate Dynamics and Interest Rate Derivative Pricing

The term structure modeling literature builds upon the insights[7] of derivative pricing, as pioneered by Black and Scholes (1973). However, it has evolved largely independently from the mainstream derivative pricing literature, due to some complexities that justify a separate treatment. First, unlike stocks that are in principle infinitely lived, the market value of a ZCB converges to its known face value at the fixed date of maturity (referred to as the pull-to-par effect). Furthermore, the price volatility has to drop to zero at maturity, while Black and Scholes (1973) make the simpler assumption that the stock volatility is constant. Second, although continuous portfolio rebalancing and, correspondingly, continuous trading is key to set up a replicating portfolio to offset the risk involved in both strands of literature, we simply cannot go in the market and buy the underlying interest rate. Unlike a stock, an interest rate is not an asset that can be traded (though a ZCB is), and hence the typical dynamic replicating strategy is not straightforward and will require an additional assumption (see below). Third, it would be inconsistent to assume that bond prices obey random dynamics while the short term interest rates are assumed to be constant. The same assumption is less troublesome in the stock derivative pricing literature, since stock price volatility typically exceeds interest rate volatility in a substantial way. Fourth, the true underlying of a ZCB is not an interest rate of a particular maturity, but rather a whole range of interest rates of different maturities.[8] We cannot limit ourselves to modeling a unique underlying ZCB or yield, we need to model the future evolution of the *entire* term structure of ZCBs. The latter feature is the key distinguishing feature of term structure models.

The pioneers of the term structure panel data approach are Vasicek (1977) and Cox et al. (1985*a*). They cope with the above complexities by modeling interest rate dynamics, instead of bond price dynamics. In that way, bond prices naturally

[7] The main insight of Black and Scholes (1973) may be summarized as follows. If we assume that the stock price follows a specific random process and that the short-term interest rate is constant, then we can always set up a portfolio of stocks and short-term bonds that replicates the payoffs of a European option. Therefore, assuming away arbitrage opportunities, the price of the European option must equal the *known* price of the replicating portfolio ("If it looks like a duck, walks like a duck, and quaks like a duck, then it must be a duck").

[8] An example clarifies the argument. If we are to value a 1-year American option on a 5-year ZCB, then we do not only need to model the 5-year ZCB, since in one year time the 5-year ZCB will have become a 4-year ZCB. Moreover, it is important to model both the 5- and 4-year ZCB in a consistent way, *i.e.* such that they do not permit arbitrage opportunities. This translates into two types of no-arbitrage conditions: (*i*) each ZCB has to be priced correctly with respect to its future ZCB price evolution and (*ii*) the prices across bonds of different maturities at each specific point in time should not imply arbitrage opportunities.

converge to par, bond price volatilities drop to zero at maturity, and the interest rate is no longer assumed to be constant. But which interest rate should we model? In the next section, it is shown that ZCB prices can be expressed in total generality as a particular conditional expectation of the face value at maturity, discounted with future *short rates*, absent arbitrage opportunities. Denote by $B(t)$ the value at time t of a bank deposit or money market account. Assume that the instantaneous return from this bank deposit, expressed in general as $dB(t)/B(t) = \mu_B(t)dt + \sigma_B(t)dW(t)$, is time-varying but completely deterministic:

$$\frac{dB(t)}{B(t)} = r(t)dt. \tag{13}$$

Integrating equation (13) and then taking the exponent leads to the following expression for $B(t)$:

$$B(t) = B(0)\exp\left(\int_0^t r(s)ds\right). \tag{14}$$

We label the expected return of the bank deposit, $\mu_B(t) = r(t)$, as the *short rate*. This bank account can be considered to be a risk free asset, given that the volatility of its return is zero, $\sigma_B(t) = 0$. For this reason, the short rate $r(t)$ is also referred to as the *risk free interest rate*. Let t denote the present time, and T the time of maturity of a ZCB with price $P(t,T)$ and continuously compounded yield $y(t,T) \equiv -\ln P(t,T)/(T-t)$. The time to maturity of this ZCB is τ, where $\tau \equiv T-t$. For modeling purposes, it is convenient to assume that at each date t ZCBs of *all* maturities τ exist. In particular, we assume that at each date t there exists a bond that matures the next instant. For this reason, the yield on that particular ZCB that matures the next instant, $r(t)$, is also referred to as the *instantaneous interest rate*. Needless to say, the short rate $r(t)$ is a theoretical construct and is not observed in reality. Nevertheless, we will use it as the basis of our term structure models. The pioneers of term structure modeling argued that the short rate is a natural *state variable*, *i.e.* a variable describing the state of the economy, in both term structure models and full-fledged macroeconomic models.

Short rate dynamics are most often modeled as a *time-homogeneous* diffusion:

$$dr(t) = \mu_r(r(t))\,dt + \sigma_r(r(t))\,dW(t). \tag{15}$$

Some researchers question the plausibility of the diffusion assumption for interest rate dynamics. First, they cast doubt on the reasonableness of the Markovian assumption. Furthermore, if the process can be assumed to be Markovian, the question remains whether or not it can be identified as a diffusion (which is more restrictive since a diffusion requires continuity). Are the discontinuities observed in the discrete data the result of the discreteness of the sampling, or rather evidence of genuine non-diffusion dynamics of the continuous-time interest rate process, like the target rate changes at US Federal Open Market Commission meetings?

These are nontrivial research questions. By nature, even if sample paths were continuous, the discretely sampled interest rate data will appear as a sequence of discrete changes. Papers that discuss this technical issue are Aït-Sahalia (2002) and Bertsimas et al. (2000). Second, do we assume diffusion dynamics or do we allow for discontinuities or jumps? Recently, models and estimation techniques have appeared that are based on jumps or point processes (Das (2000, 2002) and Piazzesi (2001)).

Modeling the short rate dynamics as in (15) allows us to take into account some of the salient features of observed interest rate dynamics. First, the Brownian motion driving process $dW(t)$ accounts for the randomness of interest rate changes. Second, Cox et al. (1985a) propose a properly scaled version of the Brownian motion to account for the heteroskedasticity of interest rates: $\sigma_r(r(t)) = \sigma\sqrt{r(t)}$. The finance literature is uniform in its view that interest rate volatility is increasing in interest rate levels, though there is some disagreement about the rate of increase (see e.g. Aït-Sahalia (1996a, 1996b), Chan et al. (1992), Chapman and Pearson (2000)). A nonparametric study of Boudoukh et al. (1998) finds that the volatility is increasing in the level of interest rates primarily for sharply upward sloping term structures. They then argue that the finding of heteroskedasticity might be due to the positively sloped average term structure, and is not a general result *per se*. Third, interest rates are more mean-reverting than a random walk would suggest. The SDE in (15) also allows to introduce *mean reversion* to the short rate dynamics, $\mu_r(r(t)) = \kappa(\theta - r(t))$. In this way, interest rates are pulled elastically towards a certain (possibly time-varying) level. Both from time series plots and introspection, it may seem obvious that interest rates must mean revert to some long-run average value. The alternative is that the variance of the short rate increases without bound as we look further into the future. The *halving-time*, *i.e.* the time it takes to mean revert to half of the current deviation from θ, can be found by computing the conditional mean of the short rate:

$$E_t[r(s)] = \theta + (r(t) - \theta)\, e^{-\kappa(s-t)}, \tag{16}$$

and by imposing that at the future time s the deviation is only half of the current deviation, $E_t[r(s)] - \theta \equiv 1/2\,(r(t) - \theta)$. Substituting the latter in (16) leads to an expression for the halving time $s - t$:

$$\textit{halving time} = s - t = \frac{\ln 2}{\kappa}. \tag{17}$$

Fourth, by restricting the parameter space (see for example the Feller condition in Cox et al. (1985a)) we can rule out negative nominal interest rates, which would be an obvious arbitrage possibility.[9]

[9] Transaction costs might occasionally impede negative nominal interest rates to lead to arbitrage opportunities. For example, in November 1998, interest rates on Japanese Treasury Bills

Given that we model the short rate in (15) as a Markov process, all that is known about future interest rates is impounded in the current short rate. Hence the value of a ZCB of any maturity may be written as a function of this instantaneous short rate and time, $P(t,T) = P(t,T,r(t))$. In this sense, ZCBs are *derivative securities*, *i.e.* securities deriving their value from the underlying short rate dynamics. As the short rate is assumed to follow a diffusion dynamics, Itô's lemma implies that ZCB prices follow diffusion dynamics as well. Running ahead of things, an arbitrage reasoning will then allow us to derive the ZCB prices as the solution to a parabolic partial differential equation, subject to a time-of-maturity boundary condition (see section 4.3).

3 Absence of Arbitrage and Its Implications

An *arbitrage opportunity* is any zero-net-investment strategy that guarantees a positive payoff in some future state of the world with no possibility of a negative payoff in all other future states of the world. Assuming the *absence of arbitrage opportunities* within a particular market will turn out to be the fundamental underlying concept that permeates this paper. The basic underlying assumptions of absence of arbitrage are individual rationality and individuals always preferring more to less (insatiability assumption). Another more contestable[10] underlying assumption is the absence of transaction and trading costs, taxes and market frictions in general.

The *law of one price* asserts that two perfect substitutes must trade at the same price. If the law would not hold and two perfect substitutes would trade at different prices, one could sell the expensive asset forward and buy the cheap one, to end up with an arbitrage opportunity. So the law of one price clearly is a direct consequence of assuming away arbitrage opportunities. However, it is not equivalent to the absence of arbitrage assumption. It is more restrictive because it deals only with the case in which two assets are identical but have different prices. It does not cover cases in which one asset, say, stochastically dominates another but may do so by different amounts in different future states of the world.

In this section, we discuss the most important implications of the absence of arbitrage. Harrison and Kreps (1979) and Harrison and Pliska (1981) formally state and prove that assuming away arbitrage opportunities within a particular market is equivalent to the existence of a so-called *pricing kernel* or price generator. They

became negative yielding an interest rate of -0.004%. Large investors found it more convenient to hold these $6m$ T-Bills as a store of value rather than holding cash because the Bills are denominated in larger amounts and can be stored electronically. See Mishkin (2001) on this.

[10] The presence of transaction costs is important and weakens the implications of absence of arbitrage by driving a wedge between what the pure absence of arbitrage would predict and what actually occurs. For example, Obstfeld and Rogoff (2000) argue that, assuming that transaction costs exist, "one can go far toward explaining a great number of the main theoretical puzzles that international macroeconomists have struggled with over 25 years".

also prove that absence of arbitrage is equivalent to the existence of a so-called *risk neutral* probability measure. These mathematical equivalents are jointly referred to as the "Fundamental Theorem of Asset Pricing". In Appendix A, it is shown that the pricing kernels of two distinct markets completely determine the exchange rate dynamics between the two corresponding currencies.

3.1 Pricing Kernel Dynamics

Whenever an economic environment precludes arbitrage, there exists a positive stochastic process or *pricing kernel*, $M(t)$, such that the product of *any* (marketed or non-marketed) security price, $V(t)$, with this pricing kernel is a $\mathcal{P}$-martingale (Harrison and Kreps (1979) and Harrison and Pliska (1981)):

$$M(t)V(t) = E_t^{\mathcal{P}}\left[M(T)V(T)\right]. \tag{18}$$

We can rewrite this as:

$$V(t) = E_t^{\mathcal{P}}\left[\frac{M(T)}{M(t)}V(T)\right]. \tag{19}$$

This basic asset pricing formula has the interpretation of an expected value under the *historical* (or *physical*, *objective*, and *data generating*) probability measure $\mathcal{P}$ of a discounted payoff, where the discounting is subject to uncertainty or randomness. The ratio $M(T)/M(t)$ is called accordingly the *stochastic discount factor* (SDF) and is identical across assets. In an exchange economy, the SDF can be interpreted as the representative agent's nominal, intertemporal, marginal rate of substitution (see Duffie (2001) or Cochrane (2001)). The pricing kernel and SDF are unique in case financial markets are complete. A *complete market* is a market in which the space of all payoffs is spanned by trading strategies in the available assets. Perfect risk elimination is possible in a complete market, meaning that, say, options can be artificially created by a suitable buy/sell-strategy in the underlying asset. This gives rise to a unique price for each derivative security, namely the initial wealth needed to finance the replicating portfolio. The issue of whether fixed income security markets are complete or incomplete is not settled and forms a fundamental empirical question (Campbell (2000)). On the one hand, factor analysis of the term structure seems to suggest that three or four factors are adequate in describing the variation in bond returns (Litterman and Scheinkman (1988), Knez et al. (1994)). On the other hand, a number of recent papers provide support for the hypothesis that fixed income security markets are incomplete by documenting security-specific or liquidity-related anomalies in the pricing of fixed income securities (see Heidari and Wu (2001)).

If markets are *incomplete*, there exists an infinite number of pricing kernels (SDFs) that generates asset prices. More specific, any $\hat{M}(t) = M(t)U(t)$, where $U(t)$ is a martingale orthogonal to both $M(t)$ and $V(t)$, also prices $V(t)$, since

$$E_t^{\mathcal{P}}\left[\left(\hat{M}(s)/\hat{M}(t)\right)V(s)\right] = E_t^{\mathcal{P}}\left[\left(M(s)/M(t)\right)V(s)\right]E_t^{\mathcal{P}}\left[U(s)/U(t)\right]$$

$$
\begin{aligned}
&= E_t^{\mathcal{P}}\left[\left(M(s)/M(t)\right)V(s)\right] \qquad (20)\\
&= V(t).
\end{aligned}
$$

The price at time t of a ZCB that matures at T, *i.e.* $P(t,T)$, yielding 1 unit of account (or 100% of the face value) at maturity, is a special case of the general equation (19):

$$
P(t,T) = E_t^{\mathcal{P}}\left[\frac{M(T)}{M(t)}P(T,T)\right] = E_t^{\mathcal{P}}\left[\frac{M(T)}{M(t)}\right]. \qquad (21)
$$

If we state how $M(t)$ evolves between t and T, we are able to price at time t the bond that matures at time T.[11] The equation makes explicit that a model to price ZCBs consists of a model for the pricing kernel dynamics. Furthermore, any diffusion is completely determined by its drift and diffusion. By applying the pricing kernel definition to the bank deposit in equation (14) and to ZCBs of different maturities, we can show that, when we assume instantaneous ZCB returns to follow a geometric Brownian motion[12]:

$$
\frac{dP(t,T)}{P(t,T)} = \mu_P(t)\,dt + \sigma_P(t)\,dW(t), \qquad (22)
$$

no-arbitrage imposes the drift of the pricing kernel dynamics to be the negative of the short rate, *i.e.* $\mu_M(t) = -r(t)$, and the diffusion to be the negative of the so-called market price of risk, *i.e.* $\sigma_M(t) = -\lambda(t)$:

$$
\frac{dM(t)}{M(t)} = -r(t)dt - \lambda(t)dW(t), \qquad (23)
$$

where $\lambda(t)$ is defined as the *market price of (interest rate) risk* and summarizes the relation between risk and expected return[13]:

$$
\lambda(t) \equiv \frac{\mu_P(t) - r(t)}{\sigma_P(t)}. \qquad (24)
$$

[11] In this way, it is unnecessary to assume a representative agent production economy. Though we take the pricing kernel dynamics as the basis for our analysis, these dynamics are always embedded in a general equilibrium model and the preferences and production technologies might be (non-trivially) reverse-engineered from it. Harrison and Kreps (1979) state that there exists an equilibrium which supports any admissible pricing kernel. As such, any pricing kernel model can be derived in a general equilibrium framework. Furthermore, since the general equilibrium requires a wide variety of assumptions about investor preferences, production technologies, and budget constraints, there may be multiple equilibria which support a given prespecified pricing kernel.

[12] As discussed above, to assume that $P(t,T)$ is a diffusion, we only have to assume that its underlying is a diffusion and that $P(t,T)$ is twice differentiable in the underlying such that Itô's Lemma holds. Indeed, a twice differentiable function of a diffusion is again a diffusion.

[13] The market price of interest rate risk reflects both the price of *real interest rate risk* and the market price of *inflation risk*. Although of relevance for bondholders, this dissertation will not be concerned with the relative contribution of both components. Little empirical work has been done yet on this important dichotomy (for a recent attempt, see Buraschi and Jiltsov (2002)).

Equations (23)-(24) can be derived informally as follows. Assume the bond dynamics to obey (22) and write the pricing kernel dynamics in general as:

$$\frac{dM(t)}{M(t)} = \mu_M(t)dt + \sigma_M(t)dW(t), \tag{25}$$

where $\mu_M(t)$ and $\sigma_M(t)$ are unknown and are to be derived from arbitrage considerations. Recall now that $M(t)P(t,T)$ is a $\mathcal{P}$-martingale process and hence:

$$E_t^{\mathcal{P}}\left[d\left(M(t)P(t,T)\right)\right] = 0. \tag{26}$$

Applying Itô's lemma to the expression between brackets in (26) yields:

$$\mu_P(t) + \mu_M(t) = -\sigma_M(t)\,\sigma_P(t)\,. \tag{27}$$

This equation should hold for any ZCB imaginable. When we apply it to the above defined bank deposit $B(t)$ (see (14)), $\mu_P(t) = \mu_B(t) = r(t)$ and $\sigma_P(t) = \sigma_B(t) = 0$, it follows that:

$$\mu_M(t) = -r(t). \tag{28}$$

Note also that, combining equations (27) and (28), it follows that the instantaneous expected return of an asset is equal to the risk free interest rate when the diffusion of that asset equals zero, or[14]:

$$\sigma_P(t) = 0 \Rightarrow \mu_P(t) = r(t). \tag{29}$$

We can also derive an expression for the pricing kernel diffusion $\sigma_M(t)$. Using two ZCBs A and B, differing in their time of maturity ($T_A \neq T_B$) with prices $P_A = P(t,T_A)$ and $P_B = P(t,T_B)$, we can always construct a portfolio of the ZCBs A and B with weights $(\beta, 1-\beta)$ that is risk free by picking the portfolio weight β such that $\beta\sigma_{P_A}(t) + (1-\beta)\,\sigma_{P_B}(t) = 0$, or equivalently $\beta = \sigma_{P_B}(t)\,/\,(\sigma_{P_B}(t) - \sigma_{P_A}(t))$. Ruling out arbitrage, it follows from equation (29) that the return of the resulting portfolio should be equal to the risk free rate:

$$\beta\mu_{P_A}(t) + (1-\beta)\,\mu_{P_B}(t) = r(t)\,. \tag{30}$$

Substituting the carefully picked weight β into equation (30) allows us to rewrite the no-arbitrage condition (30) as follows:

$$\frac{\mu_{P_A}(t) - r(t)}{\sigma_{P_A}(t)} = \frac{\mu_{P_B}(t) - r(t)}{\sigma_{P_B}(t)}. \tag{31}$$

[14] Note also that risk neutrality, $\sigma_M(t) = 0$, gives rise to an instantaneous risk free return:

$$\sigma_M(t) = 0 \Rightarrow \mu_P(t) = r(t).$$

So we find that in an arbitrage free economy, the excess return per unit of risk should be *the same* for all ZCBs and *independent of the time to maturity.* We will define this ratio that follows from no-arbitrage by the market price of risk, $\lambda(t)$:

$$\lambda(t) \equiv \frac{\mu_P(t) - r(t)}{\sigma_P(t)}. \tag{32}$$

Term structure models solve for all bond prices relative to each other. The only way to tie down the prices is by invoking this exogenous parameter, the market price of risk. The market price of risk may be interpreted as the instantaneous Sharpe ratio for holding a particular bond. Substituting (28) and (32) into (27) and solving for $\sigma_M(t)$ results in:

$$\sigma_M(t) = -\lambda(t). \tag{33}$$

Hence, we have derived equation (23) (and equation (24)), namely that the drift of the pricing kernel dynamics is the negative of the short rate, *i.e.* $\mu_M(t) = -r(t)$, and the diffusion is the negative of the so-called market price of risk-, *i.e.* $\sigma_M(t) = -\lambda(t)$. In sum, ZCB prices are determined completely by the short rate, $r(t)$, and market price of risk, $\lambda(t)$.

A term structure model imposes constraints on the bond price volatility, $\sigma_P(t)$, and this constraint, when coupled with a specification of the market price of risk, $\lambda(t)$, has implications for the expected instantaneous excess return of the bond:

$$\begin{aligned} \mu_P(t) - r(t) &= \lambda(t)\,\sigma_P(t)\,, \\ &= -Cov\left(dP(t,T)/P(t,T), dM(t)/M(t)\right). \end{aligned} \tag{34}$$

Equation (34) teaches us that risk premia originate from covariation with the pricing kernel. When the asset return dynamics covaries negatively with the pricing kernel dynamics, the risk premium is positive, and *vice versa.* The minus sign in (34) means that investors are willing to accept an expected rate of return below the risk free rate on securities which tend to have high payoffs when the pricing kernel (or marginal utility of consumption in a general equilibrium framework) is higher. This negative risk premium can be considered to be an insurance fee. An investor will demand and receive a higher compensation for an asset that behaves cyclically than for an asset that behaves countercyclically (like insurance), since the former has higher risk (in terms of consumption variability). This is a result that is already derived in the APT of Ross (1976). Basically, this relation tells you that an investor expects to get rewarded for taking on risk and that an investor is expected to pay (a fee) to shift risk away.[15]

[15] It turns out that the cash flows generated by stocks are quite likely to change systematically when the state of the economy is changing. Dividends and capital gains tend to be high in a business cycle expansion. Conversely, in a recession, they are relatively low. Given that the pricing kernel moves countercyclically with wealth, the cash flows associated with a stock tend to move in the opposite direction as the pricing kernel. The covariance in the expression for the risk premium is negative. Risk averse investors demand a higher return on stocks than a risk neutral investor would, since they are very sensitive to the risks associated with these securities.

3.2 Risk Neutral Pricing

There is another well-established equivalent approach to conduct modern finance research: *risk neutral pricing.* The "Fundamental Theorem of Finance" also states that any security price, when scaled by the money market account (see (13) on page 9), is a martingale under a specific, artificially constructed probability measure $\mathcal{Q}$ (for this reason $\mathcal{Q}$ is often referred to as a *martingale* probability measure). This holds for all assets and thus also for a ZCB. According to the theorem, $P(t,T)/B(t)$ is a $\mathcal{Q}$-martingale, or:

$$\frac{P(t,T)}{B(t)} = E_t^{\mathcal{Q}}\left[\frac{P(T,T)}{B(T)}\right]. \tag{35}$$

The existence of $\mathcal{Q}$ such that this result holds is formally proven in Harrison and Kreps (1979) and Harrison and Pliska (1981). The probability measure $\mathcal{Q}$ is unique in case markets are complete. Formally, this change of probability between the true and risk neutral measure is defined by the *Radon-Nikodym derivative* of $\mathcal{Q}$ with respect to $\mathcal{P}$:[16]

$$\frac{d\mathcal{Q}}{d\mathcal{P}} \equiv \exp\left(-\frac{1}{2}\int_t^T \lambda(s)^2 ds - \int_t^T \lambda(s)dW(s)\right). \tag{36}$$

We can rewrite (35), using the bank deposit definition (14) and the fact that $P(T,T)=1$, as:

$$P(t,T) = E_t^{\mathcal{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right], \tag{37}$$

which implies that the ZCB price is the expected value under $\mathcal{Q}$ of the face value continuously discounted back in time with the risk free rate. Equation (37) explains why $\mathcal{Q}$ is referred to as the *risk neutral* measure and the approach as *risk neutral* pricing. Every ZCB earns a risk free return under this (risk neutral) measure. Under $\mathcal{Q}$ it is *as if* we are valuing the security as a risk neutral investor. All of the investors' attitudes about the riskiness of future discount factors (and possibly future cash flows) are hidden in the transformation from $\mathcal{P}$ to $\mathcal{Q}$.

Equation (37) determines the ZCB price completely. It tells you that prices of ZCBs only depend on the distribution of the short rate $r(t)$ under $\mathcal{Q}$.[17] Previously,

[16] This special process $d\mathcal{Q}/d\mathcal{P}$ can be proven to be a martingale process and it has to satisfy the so-called *Novikov condition*, $E\left[\exp\left(\int_t^T \lambda(s)^2 ds\right)\right] < \infty$, implying that the variation in $\lambda(t)$ must be finite. Also, for $d\mathcal{Q}/d\mathcal{P}$ to exist, $\mathcal{P}$ and $\mathcal{Q}$ should be *equivalent* measures. Intuitively, what this means is that events that cannot occur in the first place, can *not* be made possible by simply changing the probability measure from one to the other. Likewise, events that *can* occur in the first place, cannot be made impossible by changing the probability measure.

[17] So, in valuing ZCBs we do not have to care about the actual interest rate dynamics under $\mathcal{P}$ or the ZCB price drift under $\mathcal{P}$. We will be concerned with $\mathcal{P}$ only when trying to account for the real world dynamics of the interest rates. This measure $\mathcal{P}$ is often used to model an underlying

we have seen that $r(t)$ and $\lambda(t)$ are the basic ingredients of the pricing kernel approach. Given the equivalence between the two approaches, the market price of risk $\lambda(t)$ has to determine the probability shift from $\mathcal{P}$ to $\mathcal{Q}$. This implicit link between the market price of risk $\lambda(t)$ and the jump from $\mathcal{P}$ to $\mathcal{Q}$ is made explicit by the Radon-Nikodym derivative and the Girsanov-Cameron-Martin theorem. The latter shows that the change in probability measure can be considered to imply a change in the drift (leaving the volatility unchanged) of the Wiener dynamics, or:

$$
\begin{aligned}
W^{\mathcal{P}}(t) &= W^{\mathcal{Q}}(t) - \int_0^t \lambda(s)ds, \\
dW^{\mathcal{P}}(t) &= dW^{\mathcal{Q}}(t) - \lambda(t)dt,
\end{aligned}
\tag{38}
$$

where it holds that $E_t^{\mathcal{Q}}\left[dW^{\mathcal{Q}}(t)\right] = 0$ and $E_t^{\mathcal{P}}\left[dW^{\mathcal{P}}(t)\right] = 0$.

We have already established that the expected return of an asset (under the data generating probability measure $\mathcal{P}$) equals the risk free rate plus an expected excess return or risk premium (equation (34)). Finance people construct an artificial risk neutral probability measure $\mathcal{Q}$ such that you get rid of this risk premium (in expected value). The change in measure implies a change in drift. The fundamental theorem says we can always find such a $\mathcal{Q}$ whenever there are no arbitrage opportunities in the economy. Under $\mathcal{Q}$, it is as if we were a risk neutral investor and the solution to the valuation problem simplifies to a discounting exercise where the risk free or short rate is used as the discount rate.

Finally, we can now show how Itô's lemma naturally brings us from the pricing kernel dynamics in equation (23) to the risk-neutral valuation expression (37). Indeed, given the $dM(t)/M(t)$ dynamics expressed in (23), an application of Itô's lemma to $d\ln M(t)$ yields:

$$
d\ln M(t) = \left(-r(t) - \tfrac{1}{2}\lambda^2(t)\right) dt - \lambda(t)dW(t). \tag{39}
$$

Integrating (39) from t until T and then taking the exponential results in:

$$
\frac{M(T)}{M(t)} = \exp\left(-\int_t^T r(s)ds - \tfrac{1}{2}\int_t^T \lambda(s)^2 ds - \int_t^T \lambda(s)dW(s)\right). \tag{40}
$$

asset (the interest rate dynamics), while the measure $\mathcal{Q}$ is used to price interest rate derivatives (ZCB prices). Of course, if we are interested in both the cross-sectional and time-series dimension of term structure models, we will need to specify $\lambda(t)$ as well.

Substituting this formula in the ZCB pricing equation (21) above, we get:

$$P(t,T) = E_t^{\mathcal{P}}\left[\underbrace{\exp\left(-\frac{1}{2}\int_t^T \lambda(s)^2 ds - \int_t^T \lambda(s)dW(s)\right)}_{\frac{d\mathcal{Q}}{d\mathcal{P}}} \exp\left(-\int_t^T r(s)ds\right)\right] \quad (41)$$

This explains the specification of the special process for the Radon-Nikodym derivative in equation (36). Substituting (36) finally leads to the risk neutral $\mathcal{Q}$-formula:

$$P(t,T) = E_t^{\mathcal{P}}\left[\frac{d\mathcal{Q}}{d\mathcal{P}} \exp\left(-\int_t^T r(s)ds\right)\right] \quad (42)$$

$$= E_t^{\mathcal{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right], \quad (43)$$

where the last step is a continuous state space extension of the more intuitive discrete state space change of measure (see appendix C in Duffie (2001) for a proof).

4 A Bird's Eye View of Term Structure Models

Term structure models abound and are typically categorized on the basis of the nature of their equilibrium, the number of state variables, and their empirical tractability. In this section, we briefly sketch these (possibly overlapping) categorizations. Our main focus will be on the strengths and weaknesses of the empirically tractable class of term structure models.

However, encompassing all these term structure model categorizations is the choice for a *continuous time* or *discrete time* framework. In the literature, continuous time models prevail for a number of reasons. First, theorists prefer the continuous-time setting for reasons of tractability. One can work with differential equations, rather than with difference equations. Moreover, Itô calculus allows for the explicit and simple computation of any twice differentiable nonlinear transformation of diffusions. Second, diffusions are attractive for empirical researchers because they are fully characterized by their instantaneous conditional mean and variance. Finally, the continuous time setting also plays a conceptual role. Since Black and Scholes (1973), many asset pricing and portfolio choice models have assumed dynamic trading in continuous time. This assumption often allows markets to be complete and hence derivative payoffs or consumption trajectories to be spanned, even when there exists an infinite number of possible states of nature and only a few traded securities.

4.1 Absolute versus Relative Pricing or General versus Partial Equilibrium

One can categorize the different term structure models according to the character of the equilibrium. Although most financial models are pure exchange models and are not concerned with production economies, it is important to ensure that there is no inconsistency between the asset return processes that are assumed and the underlying production economy. Cox et al. (1985*a*), Longstaff and Schwartz (1992*a*, 1992*b*), Nielsen and Saá-Requejo (1992), and Longstaff (1989) all develop complete or *general equilibrium* theories of the term structure. They take as given the exogenous specifications of the national (or international) economy, such as tastes, endowments, productive opportunities, and beliefs about possible future states of the economy. Then they endogenously derive the prices of ZCBs of different maturities from these exogenous specifications. Anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption all end up playing a role in determining the term structure, in a way that is consistent with maximizing behavior and rational expectations. It turns out to be difficult to derive closed-form results for arbitrary stochastic processes. This general equilibrium approach is also referred to as *absolute pricing*. In absolute pricing, each asset is priced by measuring its exposure to fundamental sources of macroeconomic risk.

Relative pricing is less ambitious and tries to price assets *given* the presumably correct prices of related assets. The underlying risk factors, the determinants of interest rates and the impact of changing interest rates on the macroeconomy are not analyzed. Most theories of the term structure fall within this framework and are *partial equilibrium* in nature. The no-arbitrage approach to the theory of the term structure is intended to explain the arbitrage-free pricing of ZCBs of different maturities and is pioneered by Vasicek (1977), Brennan and Schwartz (1979) and Dothan (1978). Assuming a stochastic evolution of one or more state variables (risk factors), we then derive the prices of all ZCBs by imposing no-arbitrage. Even with these less ambitious partial equilibrium models, the issue of consistency arises. This amounts to a requirement that assets be priced consistently with respect to each other. This requirement can be stated as an arbitrage condition within the model: the pricing of different assets should imply the same market price of risk. Again, it turns out to be difficult to derive closed-form results for arbitrary stochastic processes.

Note that in general equilibrium models (such as the capital asset pricing model), the appropriate measure of security risk is *not* its standard deviation, but the extent to which the security adds to the standard deviation of a well-diversified portfolio (or, the extent to which it adds to the systematic risk). The relative pricing framework takes the general equilibrium as given and as fully reflected in the price of the underlying asset. Risk is measured in isolation and the volatility is the only measure of risk recognized within derivative pricing.

4.2 Single- versus Multi-factor Models

With respect to term structure modeling, one can either assume that the complete yield curve is determined by a finite number of Markovian ”state variables” or ”stochastic factors”, and model the evolution of these factors, *or* one can directly model the evolution of the *entire* yield curve. In this paper, we consider exclusively the former approach. Ho and Lee (1986) are the pioneers of the latter approach in discrete-time. The corresponding continuous-time version emerged with the work of Hull and White (1990), Black et al. (1990), and Heath et al. (1992). The major advantage of their approach is that the model is tailored to fit the current market data perfectly. If the aim is to *price derivative securities*, a perfect fit of the yield and volatility curve is of paramount importance. The disadvantage is that supposedly constant model parameters are to be re-estimated continuously as market information arrives. In this sense, the data point to a misspecification in the model at each point in time. If the aim is to *explain the term structure movements* in a consistent way, the finite factor approach is better suited. In the end, both classes of models have their relative weaknesses and strengths and must be applied with care. Recently, attempts are being made to link both classes into one encompassing model (Brandt and Yaron (2001)).

Factor term structure models compress the huge amount of cross-section and time series ZCB yield information into the behavior of one or more unobserved or observed factors. The dynamics of the factors determine the shape of the yield curve at each given time point (cross-sectional dimension) and the movement of each maturity-specific yield through time (time series dimension). Typically, there exist significant correlations among yields of ZCBs with differing maturities. This suggests that a limited amount of factors might be able to capture the dynamics of the entire yield curve. The exact amount of factors will be determined by the data.

In *single-factor* models of the term structure, the whole term structure may be inferred from the level of the single factor, which is traditionally taken to be the short rate. Nowadays, single-factor models are prone to serious criticisms. First, the single-factor modeling assumption implies that changes in the yield curve and hence bond returns are (instantaneously) *perfectly* correlated across maturities. Not surprisingly, this assumption is easily contradicted by the empirical evidence. Second, the shape of the yield curve is severely restricted. Specifically, the single-factor models can only accommodate yield curves that are monotonically increasing, monotonically decreasing, or normally humped (*i.e.* $\frown$-shaped), for a given day. An inversely humped (*i.e.* $\smile$-shaped) or any other yield curve cannot be generated. Third, one-factor time-invariant parameter models tend to provide a relatively poor fit to the actual yield curve observed in the market (see e.g. Dewachter and Maes (2000) for a comparison of one-factor versus two- and three-factor model fits). More extensive empirical evidence (see e.g. Dai and Singleton (2000)) points strongly towards multifactor extensions.

A step towards a more realistic approach to the relative pricing of ZCBs of different maturities would be to allow for more underlying factors, leading to the *multi-factor* models of the term structure. In this class of models, the short rate is written as a sum of K state variables, implying that the ZCB prices are the product of single-factor bond prices:

$$r(t) = r\left(X(t)\right) = \sum_{i=1}^{K} X_i(t). \tag{44}$$

This approach facilitates the model's empirical analysis. For examples of models that follow this approach, see Langetieg (1980), Chen and Scott (1992), Longstaff and Schwartz (1992a), Sun (1992), Pearson and Sun (1994), Dai and Singleton (2000, 2002a), Dewachter and Maes (2000, 2001) and Cassola and Luis (2001). A review of the empirical properties of the term structure (see e.g. Dai and Singleton (2000)) shows the substantial improvements of fit offered by these multi-factor models. For example, short rate changes need not depend only on the current short rate level, but possibly also on other unobservable factors (a time-varying stochastic long-term mean, stochastic volatility, *etc.*). Also (linear combinations of) observable yields or macroeconomic variables can be taken to be the factors (see for example Duffie and Kan (1996), Berardi (2001), and Dewachter et al. (2001a, 2002).

For examples of models in which the factors have specific meanings, see Brennan and Schwartz (1979, short-term interest rate and consol rate), Richard (1978, real rate and expected inflation), Longstaff and Schwartz (1992, short-term interest rate and volatility of interest rate), Schaefer and Schwartz (1984, long-term interest rate and the spread between the long-term and short-term interest rates), Cox et al. (1985a, inflation rate and short-term interest rate), Andersen and Lund (1997, short-term interest rate and volatility of interest rate), Balduzzi et al. (1996, three factors; 2000, short rate and stochastic mean), Berardi (2001, output gap and inflation), and Dewachter et al. (2001a, 2002, output gap, inflation, central tendency of output gap and inflation, and real interest rate). In general, generalization to multiple factors requires care if the factors are given a specific economic interpretation. The relationship between the factors has to be taken into account explicitly, in order to be consistent with the no-arbitrage requirement. For example, if the two factors are a short-term and a long-term interest rate, it should be borne in mind that the long-term bond is related to a series of short-term bonds in a risk-neutral world, and hence the bond price movements of the long and short end have to be tied together in some way.

4.3 Affine Term Structure Models

The purpose of a multi-factor TSM is to provide a consistent (*i.e.*, arbitrage-free) explanation for the dynamics of the term structure. We discuss the popular class of affine TSMs and the inevitable trade-off between empirical flexibility and theoretical rigor in choosing a model from this class.

4.3.1 Solving the Term Structure partial differential equation

The partial differential equation that every ZCB price needs to satisfy is derived from basically three fundamental assumptions.
(i) ZCB prices are functionally related[18] to one or more, say K, stochastic factors $X(t)$:

$$P(t,T) = P(t,T,X(t)). \tag{45}$$

(ii) The underlying factors follow time-homogeneous diffusions:

$$dX(t) = \mu_X^{\mathcal{P}}(X(t))\,dt + \sigma_X(X(t))\,dW^{\mathcal{P}}(t). \tag{46}$$

(iii) Arbitrage opportunities are absent. The assumption of no arbitrage opportunities in our economy leads to (23), rewritten here in a multivariate setting:

$$\frac{dM(t)}{M(t)} = -r(t)\,dt - \Lambda(t)'\,dW(t). \tag{47}$$

Consider now a particular bond (fix the time of maturity T). In the absence of arbitrage, the drift and diffusion of any ZCB price dynamics are tied together:

$$\mu_P^{\mathcal{P}}(t) - r(t) = \sigma_P(t)'\Lambda(t), \tag{48}$$

where each ith component of $\Lambda(t)$ is the market price of risk associated with the ith risk factor, since it gives the risk premium required per unit of volatility associated with the ith source of risk $W_i(t)$. Equation (48) is the multivariate equivalent of equation (24). If $X(t)$ follows diffusion dynamics, then $P(t,T,X(t))$ will follow diffusion dynamics as well, provided the function $P(t,T,X(t))$ can be differentiated twice. An expression for the instantaneous holding return $dP(t,T)/P(t,T)$ can be derived using Itô's lemma:

$$\frac{dP(t,T)}{P(t,T)} = \mu_P^{\mathcal{P}}(t)\,dt + \sigma_P(t)\,dW^{\mathcal{P}}(t), \tag{49}$$

where:

$$\begin{cases} \mu_P^{\mathcal{P}}(t) = \dfrac{1}{P(t,T)}\left(\mu_X^{\mathcal{P}}(t)'\dfrac{\partial P}{\partial X} + \dfrac{\partial P}{\partial t} + \dfrac{1}{2}Tr\left[\sigma_X(t)\,\sigma_X(t)'\dfrac{\partial^2 P}{\partial X\partial X'}\right]\right), \\ \sigma_P(t) = \dfrac{1}{P(t,T)}\left(\dfrac{\partial P'}{\partial X}\sigma_X(t)\right). \end{cases} \tag{50}$$

[18] It is important to realize that we do not assume that the relationship between $P(t,T)$ and the factor(s) is known. On the contrary, the entire purpose of the following is deriving that function endogenously from the above assumptions, especially from the assumptions about absence of arbitrage.

Substituting (50) into (48) and reorganizing somewhat, we get a parabolic partial differential equation (PDE):

$$\frac{\partial P}{\partial t} + \mu_X^{\mathcal{Q}}(t)' \frac{\partial P}{\partial X} + \frac{1}{2} Tr \left[\sigma_X(t) \sigma_X(t)' \frac{\partial^2 P}{\partial X \partial X'} \right] = r(t) P(t,T), \tag{51}$$

with $\mu_X^{\mathcal{Q}}(t)$ defined as:

$$\mu_X^{\mathcal{Q}}(t) = \mu_X^{\mathcal{P}}(t) - \sigma_X(t)' \Lambda(t). \tag{52}$$

Equation (51) is often referred to as the *term structure PDE*. There is an important difference between the term structure PDE (51) and the Black and Scholes (1973) PDE. The term structure equation (51) is not preference-free unlike the Black and Scholes (1973) formulation (given the presence of the market price of risk parameter, $\Lambda(t)$). The reason for this is not hard to see. The Black and Scholes model takes the stock price, which reflects the market price of risk, as given. In contrast, the factor term structure model solves for all bond prices relative to each other. The only way to tie down the prices is by invoking the exogenous parameter, the market price of risk. $P(t,T)$ is the solution to this parabolic PDE, taking into account the boundary condition for the payment to be received at maturity, $P(T,T) = 1$.[19]

For a given time of maturity T, we can replace $\partial P/\partial t$ in (51) with $-\partial P/\partial \tau$. Duffie and Kan (1996) show that further restrictions on the dynamics of the state vector under $\mathcal{Q}$ must be imposed for the PDE (51) to yield a *tractable* ZCB price solution. More specifically, it is required that $r(t)$, $\mu_X^{\mathcal{Q}}(t)$, and $\sigma_X(t)\sigma_X(t)'$ are all affine functions of $X(t)$.

They define an *affine*[20] term structure model (ATSM) as a TSM where prices (yields) are exponential-affine (affine) functions of the state variables $X(t)$:

$$\begin{cases} P(X(t), \tau) = \exp\left(-A(\tau) - B(\tau)' X(t)\right), \\ y(X(t), \tau) = \dfrac{A(\tau)}{\tau} + \dfrac{B(\tau)'}{\tau} X(t), \end{cases} \tag{53}$$

where $A(\tau)$ is a scalar and $B(\tau)$ is a $K \times 1$ vector of complex functions of drift

[19] In addition to valuing ZCBs and determining the term structure, these models can handle any other interest rate derivative problem. Basically, the only thing that changes is the boundary condition at maturity. For example, Cox et al. (1985*a*) value European call options on interest rates. Applications to futures contracts, variable rate instruments, mortgages, loan commitments, *etc.* have also been undertaken. A recent example is the pricing of coupon bonds and swaptions (Singleton and Umantsev (2002)).

[20] A function $f(\cdot)$ is defined to be *affine* if it is constant-plus-linear in its argument (strictly speaking, *linear* would suffice). A univariate example would be: $f(x) = a + bx$, for real parameters a and b.

and diffusion parameters of the state variable dynamics.[21] Substituting (53) and affine expressions for $r(t)$, $\mu_X^{\mathcal{Q}}(t)$, and $\sigma_X(t)\,\sigma_X(t)'$ into (51) results in an affine expression at the left and right hand side of the term structure PDE. Equalizing the constants and $X(t)$-coefficients on the left and right hand side, respectively, leads to a system of ODEs which is far easier and less time-consuming to solve than the PDE in (51). $A(\tau)$ and $B(\tau)$ are the solutions to this system of possibly coupled ODEs with initial conditions $A(0) = 0$ and $B(0) = 0_K$. The resulting system of ODEs is relatively easy to solve using stable, accurate, and fast numerical integration techniques, such as the Runge Kutta $4th$ order algorithm that is used in this dissertation. For this reason, we refer to these solutions as quasi-closed-form solutions. We can find truly analytical or closed-form solutions only in some special cases (e.g. orthogonal multifactor Cox et al. (1985a) model). Researchers have put in great effort to find these tractable models since it makes it possible to price bonds of all possible maturities in a consistent way.

4.3.2 Parameterizing Affine Term Structure Models

Without loss of generality, Dai and Singleton (2000) parameterize the class of affine factor term structure models. They write the affine factor dynamics under $\mathcal{Q}$ as:

$$dX(t) = \mu_X^{\mathcal{Q}}(X(t))\,dt + \sigma_X(X(t))\,dW^{\mathcal{Q}}(t), \tag{54}$$

$$\mu_X^{\mathcal{Q}}(X(t)) \;=\; \tilde{K}\left(\tilde{\Theta} - X(t)\right), \tag{55}$$

$$\sigma_X(X(t)) \;=\; \Sigma\sqrt{R(t)}, \tag{56}$$

where $\tilde{K}$, $\tilde{\Theta}$, and Σ are $K \times K$, $K \times 1$, and $K \times K$ matrices, respectively. $R(t)$ is a diagonal $K \times K$ matrix with the ith diagonal element given by:

$$[R(t)]_{ii} = \alpha_i + \beta_i^{'} X(t), \tag{57}$$

where α_i is a scalar and β_i is a $K \times 1$ vector for each i. The short rate is also assumed to be affine in the state variables:

$$r(t) = r(X(t)) = \delta_0 + \delta' X(t), \tag{58}$$

where δ_0 is a scalar and δ is a $K \times 1$ vector. This setup results in the following system of ODEs:

$$\begin{cases} \dfrac{dA}{d\tau} = \tilde{\Theta}'\tilde{K}'B(\tau) + \dfrac{1}{2}\displaystyle\sum_{i=1}^{K}\left[\Sigma' B(\tau)\right]_i^2 \alpha_i - \delta_0, \\[2ex] \dfrac{dB}{d\tau} = -\tilde{K}'B(\tau) - \dfrac{1}{2}\displaystyle\sum_{i=1}^{K}\left[\Sigma' B(\tau)\right]_i^2 \beta_i + \delta, \end{cases} \tag{59}$$

[21] In the equation, $B(\tau)$ stands for the sensitivity of ZCB prices to changes in the factors and should not be confused with the bank account, defined on page 9. In the following, it should be obvious to which we are referring.

with initial conditions $A(0) = 0$ and $B(0) = 0_K$.[22]

Next, Dai and Singleton (2000) uniquely categorize the family of K-factor ATSMs into $K + 1$ *subfamilies* $A_U(L)$, $L = \{0, ..., K\}$, of ATSMs based on the number of factors, $U = \{0, ..., L\}$, that are present in the conditional factor variances. They show that every ATSM can be rewritten in a *canonical form* with identical econometric implications for the short rate and, hence, for bond prices.

Notice that Duffie and Kan (1996) require that the factor drift under $\mathcal{Q}$ is affine. They are silent though about the market price of risk vector $\Lambda(t)$ and the corresponding drift under $\mathcal{P}$. However, in order to empirically implement ATSMs, we need to know the dynamics of $X(t)$ under the actual measure $\mathcal{P}$. Duffee (2002) and Dai and Singleton (2002a) parametrized the market price of risk vector $\Lambda(t)$ as follows:

$$\Lambda(t) = \sqrt{R(t)}\lambda^0 + \sqrt{R^-(t)}\lambda^X X(t), \tag{60}$$

where λ^0 is an $K \times 1$ vector of constants, λ^X is a $K \times K$ matrix of constants, and the diagonal matrix $R^-(t)$ has zeros in its first U diagonal entries and $1/\left(\alpha_i + \beta_i^{'} X(t)\right)$ in entries $i = \{U + 1, ..., K\}$. Under this particular assumption for $\Lambda(t)$, the process for $X(t)$ also has an affine form under $\mathcal{P}$:

$$dX(t) = K\left(\Theta - X(t)\right)dt + \Sigma\sqrt{R(t)}dW^{\mathcal{P}}(t), \tag{61}$$

where

$$\begin{cases} K = \tilde{K} - \Sigma\Phi_0 + \Sigma\lambda^X, \\ \\ \Theta = K^{-1}\left(\tilde{K}\tilde{\Theta} + \Sigma\Phi_1\right), \end{cases} \tag{62}$$

where the ith row of the $K \times K$ matrix Φ_0 is $\left[\lambda^0\right]_i \beta_i^{'}$ and where Φ_1 is a $K \times 1$ vector whose ith element is $\left[\lambda^0\right]_i \alpha_i$. Affine factor dynamics under the physical measure allow for the explicit calculation of the conditional mean and variance of the factors. In turn, estimation techniques that rely on these two moments can be used (such as the Kalman filter QML approach).

The equations (54), (58), and (60) define the so-called class of *essentially affine* TSMs, in contrast to the more restrictive class of *completely affine* TSMs where the parameter vector λ^X is restricted to be a $K \times 1$ vector of zeros. The latter are analyzed in Duffie and Kan (1996), Dai and Singleton (2000), and Dewachter and Maes (2000, 2001), the former in Dai and Singleton (2002), Duffee (2002), and Dewachter et al. (2001a, 2002a).

4.3.3 Admissibility Conditions

Any sensible parameterization of ATSMs must be theoretically admissible and econometrically identified. Dai and Singleton (2000) discuss the parameter re-

[22] Note that the loadings are completely determined by the specification of the risk neutral dynamics of the state variables $X(t)$.

strictions that these two requirements place on the ATSMs discussed above. An ATSM is defined as *admissible* whenever the conditional factor variances $[R(t)]_{ii}$, $[R(t)]_{ii} = \alpha_i + \beta_i^{'} X(t)$, are positive for arbitrary choices of the state vector. Put differently, α_i and β_i cannot be picked arbitrarily, since $[R(t)]_{ii}$ needs to remain positive for all possible values of the state vector $X(t)$.[23] These (and other) parameter restrictions play a role of paramount importance in the ongoing search for the "correct" diffusion model of the term structure movements, as will be discussed next.

The academic literature has mainly focused attention on ATSMs, since this class of models yields quasi-closed-form bond pricing formulae and can easily handle multiple state variables. However, there are reasons to look for factor TSMs *outside* the affine class. First, all but one family of ATSMs face a trade-off between the structure of ZCB price volatility and admissible nonzero (un)conditional correlations among the state variables. This hampers their empirical performance. For the multivariate Gaussian ATSMs in the $A_0(K)$ family, there are no admissibility restrictions, but the $A_0(K)$ class is by construction unable to account for the heteroskedasticity in interest rates. Outside the $A_0(K)$ class, imposing admissibility conditions becomes necessary and the conditions become increasingly more stringent when U, the number of factors that are present in the conditional variance matrix, increases from 0 to K. The $A_K(K)$ class imposes the largest number of admissibility conditions on its parameter space, but allows for the most flexibility in modeling the (un)conditional volatility. As shown by Dai and Singleton (2000), when U equals K, this effectively implies that the factors must have nonnegative unconditional and zero conditional correlations. A second reason to look outside the class of ATSMs is the omitted nonlinearity in the data (see the results of Dai and Singleton (2000)). A third reason is that only $A_K(K)$ models within the class of ATSMs is able to ensure a strictly positive interest rate. Therefore, ATSMs cannot allow for negative unconditional correlations among the state variables *whilst* guaranteeing positivity of the nominal interest rate. While it may be a worthwhile sacrifice if the empirical performance of the model can be significantly improved and if the real probability of having negative interest rates, albeit posi-

[23] Dai and Singleton (2000) equalize the "admissibility" property of ATSMs with "well-defined bond prices". However, from their definition of admissibility, it is obvious that admissibility is only concerned with the short rate diffusion, whilst it remains silent on the short rate drift. Strictly speaking, "well-defined" (admissible) bond prices do not rule out negative short rates, and, in the case where one can hold cash, the presence of arbitrage opportunities. When a ZCB would offer a negative interest rate, a short position in this ZCB and a long position of an equal amount of money, both held until the time of maturity of the ZCB, would yield an arbitrage opportunity.

In my opinion, the *only* strictly arbitrage-free, theoretically admissible, and econometrically identified ATSM is the canonical model of the $A_K(K)$ class. All other classes are not arbitrage-free in the strictest sense. For an appreciation of the small probability of negative interest rates within the class of Gaussian models, see Rogers (1996). He shows that the small probability might be important when looking at knockout bonds and, more importantly, long maturity bonds.

tive, is small given appropriate choices of parameter estimates, some practitioners and academics alike hold strong opinions against term structure models that allow for negative interest rates.

The above discussion forces one to make the trade-off between empirical flexibility and theoretical rigor in choosing a particular ATSM. Do we favor a theoretically flawed, but empirically flexible model, or do we stick to a theoretically more correct model that might not be able to fit the data as well? Researchers choose the model that suits their purposes best. They use the most flexible TSM model when trying to explain the failure of the expectations theory, since interest rate heteroskedasticity is not key to explaining the puzzle. They choose a model from the $A_K(K)$ class if the short rate is required to be strictly positive. Of course, they tie their hands in doing so, but there is nothing conceptually wrong with choosing to explore a particular specification, of course.

4.3.4 Beyond the Affine Class of Term Structure Models: Quadratic Models

From the above, it is clear that ATSMs are not able to ensure positivity of interest rates without having to impose parameter restrictions and without loosing flexibility on the unconditional correlation structure amongst the state variables. Moreover, ATSMs are not able to capture the *non-linearities* in the dynamics of interest rates. The existence of the latter have, however, been demonstrated by Aït-Sahalia (1999) and others.

These problems are inherent to ATSMs and motivate the quest for non-affine models that do not suffer from these shortcomings. Roughly speaking, only one alternative class of models has been proposed: the *quadratic* TSMs (QTSMs), developed by Longstaff (1989), Beaglehole and Tenney (1991, 1992), and Constantinides (1992), and put in canonical form by Ahn et al. (2002) and Leippold and Wu (2001, 2002). In these models, ZCB prices are expressed as quadratic-exponential functions of the state variables and, similar to the ATSMs, the loadings can be solved analytically (for the one factor-case and for independent factor multi-factor cases) or by solving a system of ODEs (in the general multi-factor case).

QTSMs can potentially overcome the difficulties of ATSMs. They allow for strictly positive nominal interest rates without imposing restrictions on the correlation structure of state variables. The state variables in the canonical QTSM are all Gaussian. The non-negativity of the short rate is assured by the quadratic relationship between the short rate and the Gaussian (possibly negative) state variables. The interest rate and bond prices in the QTSM exhibit heteroskedastic conditional volatilities even though the state variables themselves do not exhibit this feature. The advantage of QTSMs over ATSMs, next to the nonexistent trade-off, is that in the former class all of the included state variables can contribute to the generation of stochastic volatility. The volatility of the interest rate is proportional to the level of the state variables. The QTSM does not result in less flexibility in specifying conditional correlations among the state variables since the conditional volatilities are induced by the quadratic structure rather than the processes of the state variables. In other words, they are free of the trade-off between heteroskedastic volatilities of the short rate and negative correlation among factors, while maintaining admissibility. Moreover, QTSMs have the potential to capture omitted non-linearities.. The quadratic term in bond yields adds nonlinearity to the dynamics. Within the affine class, one often uses multiple factors to generate the observed non-linearities in the interest rate dynamics. In contrast, nonlinearity is explicitly built into the quadratic model.

The major drawback of the QTSM relative to ATSMs lies in the estimation. Estimation of QTSMs is complicated because of the fact that there is not a one-to-one mapping from observed yields to the state variable vector any more. This is due to the quadratic dependence of the short rate on the state variables. Given this indeterminacy, non-linear filtering methods are called upon to estimate the model-implied state variables. Ahn, et al. (2000) circumvent this problem by using simulated method of moments (SMM) which effectively gives observable state variables through simulation. Then they use the reprojection methods proposed by Gallant and Tauchen (1998) to estimate the conditional factor mean.

The task of stating consistent conclusions about the *preferred* class of TSMs or about the extent to which modern TSMs fit economic moments is still daunting and ongoing. Ahn et al. (2002) mingle the classes of "purebred" models together, in order to optimize the empirical performance of the resulting "hybrid" model. They find that, when more than the two first conditional moments are to be fit, the pure QTSM outperforms all other models. Brandt and Chapman (2002) compared the classes of three factor models in terms of matching the *economic* moments of the yield curve and conclude similarly that a Gaussian QTSM dominates every ATSM. However, this is still early evidence and given the difference in data samples and available estimation methods, much robustness work remains to be done.

4.3.5 Joint Models of Macroeconomic and Term Structure Dynamics

A related, recently developed class of models *jointly* describes the term structure of interest rates and a selection of macroeconomic variables (see Ang and Piazzesi (2002), Ang et al. (2003), Berardi (2001), Dewachter et al. (2001*a*, 2002), Dewachter and Lyrio (2003), Evans and Marshall (2001), and Hördahl et al. (2002)). There are strong incentives to investigate the relationship between the dynamics of macroeconomic variables and the yield curve. *First*, there is strong evidence that the term structure predicts movements on macroeconomic activity. See for example Jorion and Mishkin (1991), Estrella and Hardouvelis (1991), Estrella and Mishkin (1996, 1997, 1998). The fact that a negatively sloped term structure predicts a recession is born out by the data. Every recession after the mid-1960s has been predicted by an inverted yield curve within 6 quarters of the impending recession. Only once did an inverted yield curve not result in a recession over this period. Evidence with respect to the euro area and individual member states seems less convincing (Berk and Van Bergeijk (2000)). *Second*, the traditional term structure model approach is not entirely satisfactory. The models compress the information that is contained in a panel of yield data into typically 3 *latent* factors. These factors are then labeled "level", "slope", or "curvature", according to the effect they have on the yield curve. Notwithstanding the insight that can be gained from a latent factor approach for risk management and the understanding of asset price behavior, we are still basically explaining yields with yields (Duffie and Kan (1996)). It is interesting to be able to interpret and identify what is underlying these latent factors in terms of observed or unobserved macroeconomic variables, such as output gap, inflation, or long-term expectations of output gap and inflation. Dewachter et al. (2001a, 2002) study exactly these kinds of questions within a continuous-time joint model of the macroeconomy and the yield curve. Berardi (2001) and Ang and Piazzesi (2002) also include macro-factors into a formal term structure model, but rely on some kind of approximation for the macroeconomic factors. Hence they are unable to give the unobservable factors a clear macroeconomic interpretation. Wu (2000) and Buraschi and Jiltsov (2002) take up the daunting task of developing a general equilibrium model. For the intuition on how to link the term structure of interest rates to macroeconomic variables dynamics, the reader is referred to Appendix A.

5 Conclusions

No-arbitrage term structure models are becoming increasingly important to national regulators and practitioners (see the recent ECB and Fed working papers of Cassola and Luis (2001), Brousseau (2002), Hördahl et al. (2002), and Evans and Marshall (2001)). The following four factors account for the increased interest in these models. *First*, term structure models are rooted in a framework where arbitrage opportunities are excluded in equilibrium. Given that bond markets are large and liquid, any reasonable equilibrium characterization of bond prices and yields should exclude the presence of arbitrage opportunities. Empirical or statistical models, popular with regulators and practitioners, are in general unable to impose this condition. *Second*, tremendous theoretical progress has been made the last decade in identifying and relaxing the restrictive assumptions in the extant no-arbitrage term structure models. For example, while the pioneering models of Vasicek (1977) and Cox et al. (1985) assumed a single risk factor is underlying the yield curve, correlated multi-factor models are now standard practice.[24] *Third*, the

[24] This paper does not review the empirical findings with respect to the modern term structure literature. A good review can be found in Dai and Singleton (2002a, b). Dai and Singleton present conclusive evidence for the U.S. that a particular model within the class of ATSMs is able to match all the key empirical findings reported by Fama and Bliss (1987) and Campbell and Shiller (1991) *at the maximum likelihood estimates* of their model parameters.

phenomenal increase in computing power allows the efficient panel estimation of multi-factor models on representative samples. While researchers had to restrict themselves either to the cross-section or the time-series estimation separately in the eighties and mid-nineties, panel data techniques are now widely used. Using a panel data framework leads to more efficient estimates of parameters and to more powerful specification tests. *Fourth*, interesting framework extensions have been elaborated. The joint modeling of the term structure together with the dynamics of macroeconomic variables allows for numerous monetary policy applications and insights in the working of the economy and the formation of expectations (see Dewachter et al. (2002) and Hördahl et al. (2002)). Another promising extension is the modeling of a multi-country term structure by taking the exchange rate dynamics into account, thereby allowing to study puzzling empirical international phenomena such as the forward premium puzzle (see Brandt and Santa-Clara (2002) and Dewachter and Maes (2001)).

Given that term structure models are gaining in importance for the above reasons, we review the vast literature in an intuitive way. Basically, there are two main approaches to set up a no-arbitrage term structure model (equivalently, to value ZCBs across the maturity spectrum): martingale pricing and risk neutral pricing. Whatever the approach assumed, in the end the underlying assumption is absence of arbitrage opportunities, which can be understood as a generalization of the law of one price. Absence of arbitrage can be shown to be equivalent to the existence of a pricing kernel. The pricing kernel dynamics over the life of the ZCB determine its price. It turns out that the short term interest rate determines the drift, while the market price of risk determines the volatility of the pricing kernel dynamics. Alternatively, absence of arbitrage can also be shown to be equivalent to the existence of a risk neutral probability measure. Under the risk neutral probability measure, the ZCB price is the conditional expectation of the face value discounted back in time with the risk free rate. The twist from the true to the risk neutral measure probability measure is accomplished by the Radon-Nikodym derivative which is fully characterized by the market price of risk. In sum, in both arbitrage free approaches only two ingredients are needed to set up a term structure model: the short rate and a market price of risk (vector) that summarizes the relationship between return and reward in the bond market.

By assuming that the state of the economy is well-described by factors that follow diffusion dynamics, factor-dependent expressions for prices and yields can be derived. More than two decades of research has resulted in a flow of seemingly widely-differing models. However, most extant models are recently shown to belong to the class of the so-called *affine term structure models*. The main advantage of this class of models is its tractability, translating in simple analytical or ordinary differential equation solution methods. In contrast, non-affine term structure models require more complex and time-consuming simulation-based or partial differential equation solution methods. We discuss the fundamental trade-off between empirical flexibility and theoretical rigor that applies to all models within the affine class of term structure models. It turns out that tractability is paid for with restrictive assumptions. We briefly discuss the non-affine models that try to confront these weaknesses. Recently, the class of quadratic term structure models has been proposed and seems to outperform the affine class in terms of matching the economic moments of the yield curve. However, given the lack of uniform data samples and the widely differing estimation methods, much robustness work remains to be done.

Appendix: Linking the Term Structure of Interest Rates to Output and Inflation Dynamics

We will briefly give the intuition of how macroeconomic variables can be linked to ZCB price dynamics. We use a simple discrete-time setup, based upon Ang and Piazzesi (2001). There are basically three steps to set up such a joint model. The first step consists of specifying separate models for the macro-variables and for the bond prices (yields). For simplicity, assume Gaussian vector autoregression (VAR) dynamics for the output gap g_t, inflation rate π_t, and the short-term interest rate i_t:

$$\begin{bmatrix} g_t \\ \pi_t \\ i_t \end{bmatrix} \equiv X_t = \mu + \Phi X_{t-1} + \Sigma \varepsilon_t. \tag{63}$$

Given the large liquidity of the bond markets and the large investment banks that stand ready to exploit any arbitrage opportunity that arises, it is reasonable to impose absence of arbitrage opportunities amongst the bond prices (yields) of different maturities. Simply including the yields into the above VAR system would result in two disadvantages. *First*, in that case we can only say something about the yields that are included in the VAR. Implications for yields that are not included in the VAR and for non-marketable yields cannot be drawn. *Second*, and more important, we cannot impose the absence of arbitrage amongst the different yields. The discrete-time equivalent of our no-arbitrage equation (21) is:

$$P_t^n = E_t^{\mathcal{P}} \left[m_{t+1} P_{t+1}^{n-1} \right], \tag{64}$$

where n denotes the time to maturity, $P_{t+n}^0 = 1$, and m_{t+1} stands for the stochastic discount factor (SDF, the ratio of the respective pricing kernels). We can solve equation (64) forward to find that every term structure model is equivalent to a model for the SDF dynamics (see Campbell, et al. (1997)):

$$\begin{aligned} P_t^n &= E_t^{\mathcal{P}} \left[m_{t+1} m_{t+2} \ldots m_{t+n} \right], & (65) \\ \ln m_{t+1} &= -i_{t+1} - \Lambda' \varepsilon_{t+1}, & (66) \end{aligned}$$

where Λ is assumed to be the market price of risk. Equation (66) can be compared to its continuous-time equivalent in equation (23).

The second step consists of establishing the link between the macro-model and the term structure model. First, it follows from the above set-up that the macro-factors enter into the drift of the pricing kernel. The third row of the VAR system in (63) can be interpreted as a Taylor rule equation

$$i_{t+1} = \mu_3 + \Phi_{[3,.]} X_t. \tag{67}$$

Substituting this discrete-time approximation of the short rate into (66) gives us

$$\ln m_{t+1} = -\mu_3 - \Phi_{[3,.]} X_t - \Lambda' \varepsilon_{t+1}. \tag{68}$$

Second, for reasons of tractability, we restrict attention to ATSMs

$$P_t^n = \exp\left(-A_n - B_n' X_t\right). \tag{69}$$

It follows that prices are conditionally lognormal.

Finally, step three basically solves for the yields as a function of the macro-factors, using the fact that $m_{t+1} P_{t+1}^{n-1}$ is conditionally lognormally distributed

$$\begin{aligned}
\ln P_t^n &= \ln\left(E_t\left[m_{t+1} P_{t+1}^{n-1}\right]\right) && (70)\\
&= E_t\left[\ln m_{t+1} + \ln P_{t+1}^{n-1}\right] + \frac{1}{2} Var_t\left[\ln m_{t+1} + \ln P_{t+1}^{n-1}\right] && (71)\\
&= E_t\left[\ln m_{t+1}\right] + E_t\left[\ln P_{t+1}^{n-1}\right] \\
&\quad + \frac{1}{2} Var_t\left[\ln m_{t+1}\right] + \frac{1}{2} Var_t\left[\ln P_{t+1}^{n-1}\right] + 2\frac{1}{2} Cov_t\left[\ln m_{t+1}, \ln P_{t+1}^{n-1}\right] && (72)
\end{aligned}$$

Compute each of the five terms on the right hand side

$$\begin{aligned}
\ln P_t^n &= \left(-\mu_3 - \Phi_{[3,.]} X_t\right) + \left(-A_{n-1} - B_{n-1}'\left(\mu + \Phi X_t\right)\right) + \left(\frac{1}{2}\Lambda'\Lambda\right) \\
&\quad + \left(\frac{1}{2} B_{n-1}' \Sigma\Sigma' B_{n-1}\right) + \left(B_{n-1}' \Sigma' \Lambda\right).
\end{aligned} \tag{73}$$

Re-express the left hand side and group terms at the right hand side

$$\begin{aligned}
-A_n - B_n' X_t &= \left(-\mu_3 + \frac{1}{2}\Lambda'\Lambda - A_{n-1} - B_{n-1}'\left(\mu - \Sigma'\Lambda\right) + \frac{1}{2} B_{n-1}' \Sigma\Sigma' B_{n-1}\right) \\
&\quad - \left(\Phi_{[3,.]} + B_{n-1}' \Phi\right) X_t.
\end{aligned} \tag{74}$$

Equalizing coefficients at left and right hand sides results in the following recursive system

$$\left\{\begin{aligned}
A_n &= A_{n-1} + \mu_3 - \tfrac{1}{2}\Lambda'\Lambda + B_{n-1}'\left(\mu - \Sigma'\Lambda\right) - \tfrac{1}{2} B_{n-1}' \Sigma\Sigma' B_{n-1} \\
B_n' &= \Phi_{[3,.]} + B_{n-1}' \Phi,
\end{aligned}\right. \tag{75}$$

which is the discrete-time equivalent of the ODE system in equation (59). The initial condition for the recursive system is found by rewriting i_t in two different ways

$$\frac{A_1}{1} + \frac{B_1'}{1} X_t = \mu_3 + \Phi_{[3,.]} X_{t-1}.$$

Hence

$$\begin{cases} A_1 = \mu_3 \\ \\ B_1 = \Phi'_{[3,.]}. \end{cases} \tag{76}$$

In sum, Ang and Piazzesi (2001) have found a relationship between the macro-variables and the yields

$$y_t^n \equiv -\frac{\ln P_t^n}{n} \tag{77}$$

$$= \frac{A_n}{n} + \frac{B_n'}{n} X_t, \tag{78}$$

where the coefficients A_n and B_n incorporate the no-arbitrage restrictions (unlike with ordinary least squares regressions, say). The discrete-time model fit can be evaluated easily. First, estimate the VAR system and get estimates for μ, Φ, and Σ. Then, compute the loadings A_1, B_1, A_2, B_2, *etc.* recursively (assume Λ zero for convenience). Finally, compute model-implied yields $y_t^n = A_n/n + (B_n'/n)\, X_t$, and compare them to the actually observed yields to gauge the fit. Dewachter et al. (2001a) find that this simple discrete-time Vasicek (1977) model shows a good fit at the short end, but a bad fit at the long end of the yield curve. Their conclusion is that observed macroeconomic variables fail to explain the yield curve at the long end. They show that the inclusion of unobserved variables can accommodate the yield curve behavior both at the short and long end of the yield curve, where the unobserved variables can be given a precise economic interpretation.

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